Bandits Games
and Combinatorial Problems in Statistics

Sébastien Bubeck,

Centre de Recerca Matemàtica, Barcelone
Standard prediction game

Adversary

1: CNN

2: NBC

K: ABC

Player
Standard prediction game

Adversary

1: CNN
2: NBC
K: ABC

$A \in \{1, \ldots, K\}$
Standard prediction game

Adversary

1: CNN

2: NBC

K: ABC

Player

\[ A \in \{1, \ldots, K\} \]
Standard prediction game

Adversary

\[ A \in \{1, \ldots, K\} \]

Player

1: CNN

2: NBC

K: ABC
Standard prediction game

Adversary

$\ell_1$

1: CNN

$\ell_2$

2: NBC

$\ell_K$

K: ABC

loss suffered: $\ell_A$

$A \in \{1, \ldots, K\}$
Standard prediction game

Adversary

Feedback: $\ell_1, \ldots, \ell_K$

Player

1: CNN

2: NBC

$K$: ABC

$A \in \{1, \ldots, K\}$

$R_n = \mathbb{E} \sum_{t=1}^{n} \ell_{A_t,t} - \min_{a \in \{1, \ldots, K\}} \mathbb{E} \sum_{t=1}^{n} \ell_{a,t}$.
Standard prediction game

Adversary → Sun → Player

Feedback: $\ell_1, \ldots, \ell_K$

Player

1: CNN

2: NBC

K: ABC

loss suffered: $\ell_A$

$A \in \{1, \ldots, K\}$

$$R_n = \mathbb{E} \sum_{t=1}^{n} \ell_{A_t,t} - \min_{a \in \{1, \ldots, K\}} \mathbb{E} \sum_{t=1}^{n} \ell_{a,t}.$$
Theorem (Hannan [1957])

There exists a strategy such that $R_n = o(n)$. 

Standard prediction game

Theorem (Cesa-Bianchi, Freund, Haussler, Helmbold, Schapire and Warmuth [1997])

*Hedge* satisfies

\[ R_n \leq \sqrt{\frac{n \log K}{2}}. \]

Moreover for any strategy,

\[ \sup_{\text{adversaries}} R_n \geq \sqrt{\frac{n \log K}{2}} + o(\sqrt{n \log K}). \]
Theorem (Cesa-Bianchi, Freund, Haussler, Helmbold, Schapire and Warmuth [1997])

**Hedge** satisfies

\[ R_n \leq \sqrt{\frac{n \log K}{2}}. \]

Moreover for any strategy,

\[ \sup_{\text{adversaries}} R_n \geq \sqrt{\frac{n \log K}{2}} + o(\sqrt{n \log K}). \]
Multi-armed bandit game

Adversary

Player
Multi-armed bandit game

Adversary

Player

$A \in \{1, \ldots, K\}$
Multi-armed bandit game

Adversary

Player

$A \in \{1, \ldots, K\}$
Multi-armed bandit game

Adversary

$A \in \{1, \ldots, K\}$

Player
Multi-armed bandit game

Adversary

Player

\[ A \in \{1, \ldots, K\} \]

loss suffered: \( \ell_A \)
Multi-armed bandit game

Adversary → Player

\[ A \in \{1, \ldots, K\} \]

loss suffered: \( \ell_A \)

Feedback: \( \ell_A \)
Theorem (Auer, Cesa-Bianchi, Freund and Schapire [1995])

\textbf{Exp3 satisfies:}

\[ R_n \leq \sqrt{2nK \log K}. \]

Moreover for any strategy,

\[ \sup_{\text{adversaries}} R_n \geq \frac{1}{4} \sqrt{nK} + o(\sqrt{nK}). \]
Minimax regret for the multi-armed bandit game

**Theorem (Auer, Cesa-Bianchi, Freund and Schapire [1995])**

*Exp3 satisfies:*

\[ R_n \leq \sqrt{2nK \log K}. \]

*Moreover for any strategy,*

\[ \sup_{\text{adversaries}} R_n \geq \frac{1}{4} \sqrt{nK} + o(\sqrt{nK}). \]
Minimax regret for the multi-armed bandit game

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Poly INF satisfies:

\[ R_n \leq 2\sqrt{2nK}. \]
Minimax regret for the multi-armed bandit game

Robbins [1952]
\( \ell_{a,1}, \ldots, \ell_{a,n} \text{ iid} \)

Hannan [1957]

Cesa-Bianchi et al. [1997]

Auer et al. [1995]

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Poly INF satisfies:

\[ R_n \leq 2\sqrt{2nK}. \]
Minimax regret for the multi-armed bandit game

Robbins [1952]
\( \ell_{a,1}, \ldots, \ell_{a,n} \) iid

Hannan [1957]

Lai and Robbins [1985]

Cesa-Bianchi et al. [1997]

Auer et al. [1995]

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Poly INF satisfies:

\[ R_n \leq 2\sqrt{2nK}. \]
Minimax regret for the multi-armed bandit game

\[ \ell_{a,1}, \ldots, \ell_{a,n} \text{ iid} \]

Hannan [1957]

Lai and Robbins [1985]

Cesa-Bianchi et al. [1997]

Auer et al. [1995]

Auer et al. [2002]

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Poly INF satisfies:

\[ R_n \leq 2\sqrt{2nK}. \]
Minimax regret for the multi-armed bandit game

Robbins [1952]
ℓ_a,1, . . . , ℓ_a,n iid

Hannan [1957]

Lai and Robbins [1985]

Cesa-Bianchi et al. [1997]

Auer et al. [1995]

Auer et al. [2002]

Audibert and Bubeck [2009]

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Poly INF satisfies:

\[ R_n \leq 2\sqrt{2nK}. \]
Minimax regret for the multi-armed bandit game

Robbins [1952]
\( \ell_{a,1}, \ldots, \ell_{a,n} \) iid

Hannan [1957]

Lai and Robbins [1985]

Cesa-Bianchi et al. [1997]

Auer et al. [2002]

Auer et al. [1995]

Audibert and Bubeck [2009]

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Poly INF satisfies:

\[ R_n \leq 2\sqrt{2nK}. \]
High level idea of the proof

- Start with an **Abel transform** on the regret

- Then **multivariate Taylor expansion** on the instantaneous regrets, using the **implicit function theorem**

- Control the main term in the expansion with **Hölder's inequality**

- Control the second order terms with **concentration inequalities for supermartingales**
High level idea of the proof

- Start with an Abel transform on the regret

- Then multivariate Taylor expansion on the instantaneous regrets, using the implicit function theorem

- Control the main term in the expansion with Hölder's inequality

- Control the second order terms with concentration inequalities for supermartingales
High level idea of the proof

- Start with an **Abel transform** on the regret

- Then **multivariate Taylor expansion** on the instantaneous regrets, using the **implicit function theorem**

- Control the main term in the expansion with **Hölder’s inequality**

- Control the second order terms with **concentration inequalities** for supermartingales
High level idea of the proof

- Start with an *Abel transform* on the regret

- Then *multivariate Taylor expansion* on the instantaneous regrets, using the *implicit function theorem*

- Control the main term in the expansion with *Hölder’s inequality*

- Control the second order terms with *concentration inequalities for supermartingales*
Other contributions to bandit theory

- Simple Regret
- Multi-Armed
- Continuously Armed
- Contextual Bandits
- Planning
- Stochastic Bandit
- Label Efficient
- Combinatorial Bandits
- Adversarial Bandit
- Multi-Armed
- Contextual Bandits
- Applications
- Bayesian Model
- Continuously Armed
- Contextual Bandits
- Planning
- Stochastic Bandit
- Label Efficient
- Combinatorial Bandits
- Adversarial Bandit
- Multi-Armed
- Contextual Bandits
- Applications
- Bayesian Model
Other contributions to bandit theory

- Multi-armed bandit
- Label efficient
- Adversarial bandit
Other contributions to bandit theory

- Player has to ask for the feedback
- He can ask it at most \( m \) times

Tools for the lower bound:
- Pinsker’s inequality, Fano’s lemma,
- chain rule for Kullback-Leibler divergence

Theorem (Audibert and Bubeck [2010])

Standard game: \( 0.03 \ n \sqrt{\frac{\log K}{m}} \leq \inf \sup R_n \leq n \sqrt{\frac{\log K}{2m}} \)

Bandit game: \( 0.04 \ n \sqrt{\frac{K}{m}} \leq \inf \sup R_n \leq 8 \ n \sqrt{\frac{K}{m}} \)
Other contributions to bandit theory

- Player has to ask for the feedback
- He can ask it at most \( m \) times

Tools for the lower bound:
- Pinsker’s inequality
- Fano’s lemma
- Chain rule for Kullback-Leibler divergence

Theorem (Audibert and Bubeck [2010])

\[
\text{Standard game: } 0.03 \, n \sqrt{\frac{\log K}{m}} \leq \inf \sup R_n \leq n \sqrt{\frac{\log K}{2m}}
\]

\[
\text{Bandit game: } 0.04 \, n \sqrt{\frac{K}{m}} \leq \inf \sup R_n \leq 8 \, n \sqrt{\frac{K}{m}}
\]
Other contributions to bandit theory

- Player has to ask for the feedback
- He can ask it at most $m$ times

Tools for the lower bound:
- Pinsker’s inequality, Fano’s lemma,
- chain rule for Kullback-Leibler divergence

Theorem (Audibert and Bubeck [2010])

**Standard game:** $0.03 \sqrt{n \frac{\log K}{m}} \leq \inf \sup R_n \leq n \sqrt{\frac{\log K}{2m}}$

**Bandit game:** $0.04 \sqrt{\frac{K}{m}} \leq \inf \sup R_n \leq 8 n \sqrt{\frac{K}{m}}$
Other contributions to bandit theory

- Player has to ask for the feedback
- He can ask it at most $m$ times

Tools for the lower bound:
- Pinsker’s inequality, Fano’s lemma,
- chain rule for Kullback-Leibler divergence

**Theorem (Audibert and Bubeck [2010])**

**Standard game:**
$$0.03 \; n \sqrt{\frac{\log K}{m}} \leq \inf \sup R_n \leq n \sqrt{\frac{\log K}{2m}}$$

**Bandit game:**
$$0.04 \; n \sqrt{\frac{K}{m}} \leq \inf \sup R_n \leq 8 \; n \sqrt{\frac{K}{m}}$$
Other contributions to bandit theory

Simple regret

Stochastic bandit
Other contributions to bandit theory

- Frequentist view on offline optimal learning, [Frazier and Powell, 2010]
- Bubeck, Munos and Stoltz [2009, 2010]: links between offline and online setting

**Theorem (Audibert, Bubeck and Munos [2010])**

Let $\mu_i$ be the expected loss of action $i$. Assume that there is a unique optimal action $i^*$. Let $H = \sum_{i \neq i^*} (\mu_i - \mu_{i^*})^{-2}$. Then

$$\exp \left( -c' \frac{n \log K}{H} \right) \leq \inf_{\text{Player}} \mathbb{P}(A_n \neq i^*) \leq K^2 \exp \left( -c \frac{n}{H \log K} \right).$$
Other contributions to bandit theory

- Frequentist view on offline optimal learning, [Frazier and Powell, 2010]
- Bubeck, Munos and Stoltz [2009, 2010]: links between offline and online setting

**Theorem (Audibert, Bubeck and Munos [2010])**

Let $\mu_i$ be the expected loss of action $i$. Assume that there is a unique optimal action $i^*$. Let $H = \sum_{i \neq i^*} (\mu_i - \mu_{i^*})^{-2}$. Then

$$\exp\left(-c' \frac{n \log K}{H}\right) \leq \inf_{\text{Player}} \mathbb{P}(A_n \neq i^*) \leq K^2 \exp\left(-c \frac{n}{H \log K}\right).$$
Other contributions to bandit theory

• Frequentist view on offline optimal learning, [Frazier and Powell, 2010]

• Bubeck, Munos and Stoltz [2009, 2010]: links between offline and online setting

Theorem (Audibert, Bubeck and Munos [2010])

Let $\mu_i$ be the expected loss of action $i$. Assume that there is a unique optimal action $i^*$. Let $H = \sum_{i \neq i^*} (\mu_i - \mu_{i^*})^{-2}$. Then

$$\exp\left(-c' \frac{n \log K}{H}\right) \leq \inf_{\text{Player}} \mathbb{P}(A_n \neq i^*) \leq K^2 \exp\left(-c \frac{n}{H \log K}\right).$$
Other contributions to bandit theory

Continuously armed

Stochastic bandit
Other contributions to bandit theory

- \{1, \ldots, K\} replaced by arbitrary set \( \mathcal{X} \)
- Tools: geometry in metric spaces, Hoeffding-Azuma’s inequality for martingales

**Theorem (Bubeck, Munos, Stoltz and Szepesvari [2009, 2010])**

Let \( \mathcal{X} \) be a compact subset of \( \mathbb{R}^D \) and \( \mathcal{F} \) be the set of bandits problems such that the mean-loss function is 1-Lipschitz (with respect to some norm). Then we have

\[
\inf_{\mathcal{F}} \sup R_n = \tilde{\Theta} \left( n^{\frac{D+1}{D+2}} \right).
\]
Other contributions to bandit theory

- $\{1, \ldots, K\}$ replaced by arbitrary set $\mathcal{X}$
- Tools: geometry in metric spaces, Hoeffding-Azuma’s inequality for martingales

**Theorem (Bubeck, Munos, Stoltz and Szepesvari [2009, 2010])**

Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^D$ and $\mathcal{F}$ be the set of bandits problems such that the mean-loss function is 1-Lipschitz (with respect to some norm). Then we have

$$\inf_{\mathcal{F}} \sup R_n = \tilde{\Theta} \left( n^{\frac{D+1}{D+2}} \right).$$
Other contributions to bandit theory

- \{1, \ldots, K\} replaced by arbitrary set \(\mathcal{X}\)

- Tools: geometry in metric spaces, Hoeffding-Azuma’s inequality for martingales

Theorem (Bubeck, Munos, Stoltz and Szepesvari [2009, 2010])

Let \(\mathcal{X}\) be a compact subset of \(\mathbb{R}^D\) and \(\mathcal{F}\) be the set of bandits problems such that the mean-loss function is 1-Lipschitz (with respect to some norm). Then we have

\[
\inf_{\mathcal{F}} \sup_{n} R_n = \tilde{\Theta}\left(n^{D+1 \over D+2}\right).
\]
Other contributions to bandit theory

Stochastic bandit

planning
Other contributions to bandit theory

- \{1, \ldots, K\} replaced by \{1, \ldots, K\}^*
- loss of \(t^{th}\) action discounted by \(\gamma^t\)

Theorem (Bubeck and Munos [2010])

\[
\inf \sup R_n = \begin{cases} 
\Theta \left( n^{1 - \frac{\log 1/\gamma}{\log K}} \right) & \text{if } \gamma \sqrt{K} > 1 \\
\Theta \left( \sqrt{n} \right) & \text{if } \gamma \sqrt{K} \leq 1 
\end{cases}
\]
Other contributions to bandit theory

- $\{1, \ldots, K\}$ replaced by $\{1, \ldots, K\}^*$
- loss of $t^{th}$ action discounted by $\gamma^t$

Theorem (Bubeck and Munos [2010])

$$\inf \sup R_n = \begin{cases} \tilde{\Theta} \left( n^{1 - \frac{\log \frac{1}{\gamma}}{\log K}} \right) & \text{if } \gamma \sqrt{K} > 1 \\ \tilde{\Theta} \left( \sqrt{n} \right) & \text{if } \gamma \sqrt{K} \leq 1 \end{cases}$$
Other contributions to bandit theory

- \{1, \ldots, K\} replaced by \{1, \ldots, K\}^*
- loss of \(t^{th}\) action discounted by \(\gamma^t\)

Theorem (Bubeck and Munos [2010])

\[\inf \sup R_n = \begin{cases} 
\tilde{\Theta} \left( n^{1 - \frac{\log 1/\gamma}{\log K}} \right) & \text{if } \gamma \sqrt{K} > 1 \\
\tilde{\Theta} \left( \sqrt{n} \right) & \text{if } \gamma \sqrt{K} \leq 1
\end{cases}\]
Other contributions to bandit theory

Theorem (Bubeck and Slivkins [2011])

\( SAO \) satisfies in the stochastic model: \( R_n = O(\log^2(n)) \), and in the adversarial model \( R_n = \tilde{O}(\sqrt{n}) \).
Other contributions to bandit theory

Tools: Sequential hypothesis testing, Bernstein’s inequality for martingales

Theorem (Bubeck and Slivkins [2011])

*SAO* satisfies in the stochastic model: $R_n = O(\log^2(n))$, and in the adversarial model $R_n = \tilde{O}(\sqrt{n})$. 
Other contributions to bandit theory

Tools: Sequential hypothesis testing, Bernstein’s inequality for martingales

Theorem (Bubeck and Slivkins [2011])

*SAO* satisfies in the stochastic model: $R_n = O(\log^2(n))$, and in the adversarial model $R_n = \tilde{O}(\sqrt{n})$. 
Other contributions to bandit theory

- Adversarial bandit
- Combinatorial bandits
Other contributions to bandit theory

Adversarial bandit

Combinatorial bandits
Combinatorial prediction game

Adversary

Player
Combinatorial prediction game

Adversary

Player

[Diagram of a combinatorial prediction game with nodes and edges labeled for the adversary and player.]
Combinatorial prediction game

Adversary

Player
Combinatorial prediction game

Adversary →

Player →
Combinatorial prediction game

Adversary

Player

loss suffered: $\ell_2 + \ell_7 + \ldots + \ell_d$
Combinatorial prediction game

Adversary →

Feedback:
\[
\{ \text{Full Info: } \ell_1, \ell_2, \ldots, \ell_d \}
\]

Player →

loss suffered: \( \ell_2 + \ell_7 + \ldots + \ell_d \)
Combinatorial prediction game

Adversary \rightarrow \text{Feedback: } \begin{cases} \text{Full Info: } & \ell_1, \ell_2, \ldots, \ell_d \\ \text{Semi-Bandit: } & \ell_2, \ell_7, \ldots, \ell_d \\ \text{Bandit: } & \ell_2 + \ell_7 + \ldots + \ell_d \end{cases} \\
\text{loss suffered: } \ell_2 + \ell_7 + \ldots + \ell_d

Player \rightarrow
Combinatorial prediction game

Adversary → Feedback:

\begin{align*}
\text{Full Info:} & \quad \ell_1, \ell_2, \ldots, \ell_d \\
\text{Semi-Bandit:} & \quad \ell_2, \ell_7, \ldots, \ell_d \\
\text{Bandit:} & \quad \ell_2 + \ell_7 + \ldots + \ell_d
\end{align*}

loss suffered: \( \ell_2 + \ell_7 + \ldots + \ell_d \)
Notations

\[ S \subset \{0, 1\}^d \]

\[ \ell_t \in [0, 1]^d \]

\[ V_t \in S, \text{ loss suffered: } \ell_t^T V_t. \]

**Key idea:** \( V_t \sim p_t, \ p_t \in \Delta(S) \). Then, unbiased estimate \( \tilde{\ell}_t \) of the loss \( \ell_t \):

- \( \tilde{\ell}_t = \ell_t \) in the full information game,
- \( \tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i=1} p_t(V)} V_{i,t} \) in the semi-bandit game,
- \( \tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t, \) with \( P_t = \mathbb{E}_{V \sim p_t}(VV^T) \) in the bandit game.
Notations

\[ S \subset \{0, 1\}^d \]

\[ \ell_t \in [0, 1]^d \]

\[ V_t \in S, \text{ loss suffered: } \ell_t^T V_t. \]

**Key idea:** \( V_t \sim p_t, \quad p_t \in \Delta(S) \). Then, unbiased estimate \( \tilde{\ell}_t \) of the loss \( \ell_t \):

- \( \tilde{\ell}_t = \ell_t \) in the full information game,
- \( \tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S : V_i = 1} p_t(V)} V_{i,t} \) in the semi-bandit game,
- \( \tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t \), with \( P_t = \mathbb{E}_{V \sim p_t}(VV^T) \) in the bandit game.
Key idea: $V_t \sim p_t$, $p_t \in \Delta(S)$. Then, unbiased estimate $\tilde{\ell}_t$ of the loss $\ell_t$:

- $\tilde{\ell}_t = \ell_t$ in the full information game,
- $\tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i = 1} p_t(V)} V_{i,t}$ in the semi-bandit game,
- $\tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t$, with $P_t = \mathbb{E}_{V \sim p_t} (V V^T)$ in the bandit game.
Key idea: $V_t \sim p_t$, $p_t \in \Delta(S)$. Then, unbiased estimate $\tilde{\ell}_t$ of the loss $\ell_t$:

- $\tilde{\ell}_t = \ell_t$ in the full information game,
- $\tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i = 1} p_t(V)} V_{i,t}$ in the semi-bandit game,
- $\tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t$, with $P_t = \mathbb{E}_{V \sim p_t}(VV^T)$ in the bandit game.
Key idea: $V_t \sim p_t$, $p_t \in \Delta(S)$. Then, unbiased estimate $\tilde{\ell}_t$ of the loss $\ell_t$:

- $\tilde{\ell}_t = \ell_t$ in the full information game,
- $\tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i = 1} p_t(V)} V_{i,t}$ in the semi-bandit game,
- $\tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t$, with $P_t = \mathbb{E}_{V \sim p_t}(VV^T)$ in the bandit game.
Notations

$S \subset \{0, 1\}^d$

$
\ell_t \in [0, 1]^d$

$V_t \in S$, loss suffered: $\ell_t^T V_t$.

**Key idea:** $V_t \sim p_t$, $p_t \in \Delta(S)$. Then, unbiased estimate $\tilde{\ell}_t$ of the loss $\ell_t$:

- $\tilde{\ell}_t = \ell_t$ in the full information game,
- $\tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i = 1} p_t(V)} V_{i,t}$ in the semi-bandit game,
- $\tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t$, with $P_t = \mathbb{E}_{V \sim p_t}(VV^T)$ in the bandit game.
Notations

\[ S \subset \{0, 1\}^d \]

\[ \ell_t \in [0, 1]^d \]

\[ V_t \in S, \text{ loss suffered: } \ell_t^T V_t. \]

**Key idea:** \( V_t \sim p_t, \ p_t \in \Delta(S) \). Then, unbiased estimate \( \tilde{\ell}_t \) of the loss \( \ell_t \):

- \( \tilde{\ell}_t = \ell_t \) in the full information game,
- \( \tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i = 1} p_t(V)} V_{i,t} \) in the semi-bandit game,
- \( \tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t \), with \( P_t = \mathbb{E}_{V \sim p_t}(VV^T) \) in the bandit game.
Notations

\[ S \subset \{0, 1\}^d \]

\[ \ell_t \in [0, 1]^d \]

\[ V_t \in S, \text{ loss suffered: } \ell_t^T V_t. \]

Key idea: \( V_t \sim p_t, \ p_t \in \Delta(S) \). Then, unbiased estimate \( \tilde{\ell}_t \) of the loss \( \ell_t \):

- \( \tilde{\ell}_t = \ell_t \) in the full information game,
- \( \tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in S: V_i = 1} p_t(V)} V_{i,t} \) in the semi-bandit game,
- \( \tilde{\ell}_t = P_t^+ V_t V_t^T \ell_t \), with \( P_t = \mathbb{E}_{V \sim p_t}(VV^T) \) in the bandit game.
Definition

Let $\mathcal{D}$ be a convex subset of $\mathbb{R}^d$ with nonempty interior $\text{int}(\mathcal{D})$ and boundary $\partial \mathcal{D}$. We call Legendre any function $F : \mathcal{D} \to \mathbb{R}$ such that

- $F$ is strictly convex and admits continuous first partial derivatives on $\text{int}(\mathcal{D})$,
- For any $u \in \partial \mathcal{D}$, for any $v \in \text{int}(\mathcal{D})$, we have

$$\lim_{s \to 0, s > 0} (u - v)^T \nabla F((1 - s)u + sv) = +\infty.$$
Definition

Let $\mathcal{D}$ be a convex subset of $\mathbb{R}^d$ with nonempty interior $\text{int}(\mathcal{D})$ and boundary $\partial \mathcal{D}$. We call Legendre any function $F : \mathcal{D} \rightarrow \mathbb{R}$ such that

- $F$ is strictly convex and admits continuous first partial derivatives on $\text{int}(\mathcal{D})$,
- For any $u \in \partial \mathcal{D}$, for any $v \in \text{int}(\mathcal{D})$, we have

$$\lim_{s \rightarrow 0, s > 0} (u - v)^T \nabla F((1 - s)u + sv) = +\infty.$$
Legendre function

**Definition**

Let $\mathcal{D}$ be a convex subset of $\mathbb{R}^d$ with nonempty interior $\text{int}(\mathcal{D})$ and boundary $\partial \mathcal{D}$. We call Legendre any function $F : \mathcal{D} \to \mathbb{R}$ such that

- $F$ is strictly convex and admits continuous first partial derivatives on $\text{int}(\mathcal{D})$,
- For any $u \in \partial \mathcal{D}$, for any $v \in \text{int}(\mathcal{D})$, we have

$$\lim_{s \to 0, s > 0} (u - v)^T \nabla F((1 - s)u + sv) = +\infty.$$
Bregman divergence

**Definition**

The Bregman divergence $D_F : \mathcal{D} \times \text{int}(\mathcal{D})$ associated to a Legendre function $F$ is defined by

$$D_F(u, v) = F(u) - F(v) - (u - v)^T \nabla F(v).$$
Parameter: $F$ Legendre on $\mathcal{D} \supset \text{Conv}(S)$

1. $w_{t+1}' \in \mathcal{D}$:
   $$\nabla F(w_{t+1}') = \nabla F(w_t) - \tilde{\ell}_t$$

2. $w_{t+1} \in \text{argmin}_{w \in \text{Conv}(S)} D_F(w, w_{t+1}')$

3. $p_{t+1} \in \Delta(S)$: $w_{t+1} = \mathbb{E}_{V \sim p_{t+1}} V$
Parameter: $F$ Legendre on $\mathcal{D} \supset Conv(S)$

1. $w'_{t+1} \in \mathcal{D}$:
   \[ \nabla F(w'_{t+1}) = \nabla F(w_t) - \tilde{\ell}_t \]

2. $w_{t+1} \in \text{argmin}_{w \in Conv(S)} D_F(w, w'_{t+1})$

3. $p_{t+1} \in \Delta(S)$: $w_{t+1} = \mathbb{E}_{V \sim p_{t+1}} V$
Parameter: $F$ Legendre on $D \supset Conv(S)$

1. $w'_{t+1} \in D : \nabla F(w'_{t+1}) = \nabla F(w_t) - \tilde{\ell}_t$

2. $w_{t+1} \in \arg\min_{w \in Conv(S)} D_F(w, w'_{t+1})$

3. $p_{t+1} \in \Delta(S) : w_{t+1} = \mathbb{E}_{V \sim p_{t+1}} V$
Parameter: \( F \) Legendre on \( \mathcal{D} \supset \text{Conv}(S) \)

1. \( w'_{t+1} \in \mathcal{D} : \nabla F(w'_{t+1}) = \nabla F(w_t) - \tilde{\ell}_t \)

2. \( w_{t+1} \in \arg\min_{w \in \text{Conv}(S)} D_F(w, w'_{t+1}) \)

3. \( p_{t+1} \in \Delta(S) : w_{t+1} = \mathbb{E}_{V \sim p_{t+1}} V \)
Parameter: $F$ Legendre on $\mathcal{D} \supset \text{Conv}(S)$

1. $w'_{t+1} \in \mathcal{D}$:  
   $$\nabla F(w'_{t+1}) = \nabla F(w_t) - \tilde{\ell}_t$$

2. $w_{t+1} \in \text{argmin} \ D_F(w, w'_{t+1}) \quad \forall w \in \text{Conv}(S)$

3. $p_{t+1} \in \Delta(S)$:  
   $$w_{t+1} = \mathbb{E}_{V \sim p_{t+1}} V$$
Parameter: $F$ Legendre on $\mathcal{D} \supset \text{Conv}(S)$

1. $w'_{t+1} \in \mathcal{D}$:
   \[
   \nabla F(w'_{t+1}) = \nabla F(w_t) - \tilde{\ell}_t
   \]

2. $w_{t+1} \in \arg\min_{w \in \text{Conv}(S)} D_F(w, w'_{t+1})$

3. $p_{t+1} \in \Delta(S): w_{t+1} = \mathbb{E}_{V \sim p_{t+1}} V$
General regret bound for CLEB

Theorem (Audibert, Bubeck and Lugosi [2011])

If $F$ admits a Hessian $\nabla^2 F$ always invertible then,

$$R_n \preceq \text{diam}_{DF}(S) + \mathbb{E} \sum_{t=1}^{n} \tilde{\ell}_t^T (\nabla^2 F(w_t))^{-1} \tilde{\ell}_t.$$ 

Key tool: Pythagorean theorem for Bregman divergences
Different instances of CLEB: LinExp (Entropy Function)

\[ D = [0, +\infty)^d, \quad F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i \]

\[
\begin{cases}
\text{Full Info: Hedge} \\
\text{Semi-Bandit=Bandit: Exp3}
\end{cases}
\]

\[
\begin{cases}
\text{Full Info: Component Hedge} \\
\text{Koolen, Warmuth and Kivinen [2010]}
\end{cases}
\]

\[
\begin{cases}
\text{Semi-Bandit: MW} \\
\text{Kale, Reyzin and Schapire [2010]}
\end{cases}
\]

\[
\begin{cases}
\text{Bandit: new algorithm}
\end{cases}
\]
Different instances of CLEB: LinExp (Entropy Function)

\[ \mathcal{D} = [0, +\infty)^d, \quad F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i \]

- **Full Info:** Hedge
  - Koolen, Warmuth and Kivinen [2010]
- **Semi-Bandit=Bandit:** Exp3
  - Kale, Reyzin and Schapire [2010]
  - New algorithm
- **Full Info:** Component Hedge
  - Koolen, Warmuth and Kivinen [2010]
Different instances of CLEB: LinExp (Entropy Function)

\[ \mathcal{D} = [0, +\infty)^d, \quad F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i \]

\[ \text{Full Info: Hedge} \]
\[ \text{Semi-Bandit=Bandit: Exp3} \]

\[ \text{Full Info: Component Hedge} \]
\[ \text{Koolen, Warmuth and Kivinen [2010]} \]

\[ \text{Semi-Bandit: MW} \]
\[ \text{Kale, Reyzin and Schapire [2010]} \]

\[ \text{Bandit: new algorithm} \]
Different instances of CLEB: LinExp (Entropy Function)

\[ D = [0, +\infty)^d, \quad F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i \]

\[ \begin{cases} 
\text{Full Info: Hedge} \\
\text{Semi-Bandit} = \text{Bandit: Exp3} \\
\end{cases} \]

\[ \begin{cases} 
\text{Full Info: Component Hedge} \\
\text{Koolen, Warmuth and Kivinen [2010]} \\
\end{cases} \]

\[ \begin{cases} 
\text{Semi-Bandit: MW} \\
\text{Kale, Reyzin and Schapire [2010]} \\
\text{Bandit: new algorithm} \\
\end{cases} \]
Different instances of CLEB: LinExp (Entropy Function)

\[ D = [0, +\infty)^d, \quad F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i \]

- Full Info: Hedge
- Semi-Bandit=Bandit: Exp3

Full Info: Component Hedge
Koolen, Warmuth and Kivinen [2010]

Semi-Bandit: MW
Kale, Reyzin and Schapire [2010]

Bandit: new algorithm
Different instances of CLEB: LinExp (Entropy Function)

\[ D = [0, +\infty)^d, \quad F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i \]

\[
\begin{cases}
\text{Full Info: Hedge} \\
\text{Semi-Bandit=Bandit: Exp3}
\end{cases}
\]

\[
\begin{cases}
\text{Full Info: Component Hedge} \\
\text{Koolen, Warmuth and Kivinen [2010]}
\end{cases}
\]

\[
\begin{cases}
\text{Semi-Bandit: MW} \\
\text{Kale, Reyzin and Schapire [2010]}
\end{cases}
\]

\[
\begin{cases}
\text{Bandit: new algorithm}
\end{cases}
\]
Different instances of CLEB: LinINF (Exchangeable Hessian)

\[ \mathcal{D} = [0, +\infty)^d, \ F(x) = \sum_{i=1}^{d} \int_{0}^{x_i} \psi^{-1}(s)ds \]

\{ \psi(x) = \exp(\eta x) : \text{LinExp} \\
\psi(x) = (-\eta x)^{-q}, q > 1 : \text{LinPoly} \} 

INF, Audibert and Bubeck [2009]
Different instances of CLEB: LinINF (Exchangeable Hessian)

\[ D = [0, +\infty)^d, \ F(x) = \sum_{i=1}^{d} \int_{0}^{x_i} \psi^{-1}(s)ds \]

\[ \psi(x) = \exp(\eta x) : \text{LinExp} \]
\[ \psi(x) = (-\eta x)^{-q}, q > 1 : \text{LinPoly} \]

INF, Audibert and Bubeck [2009]
Different instances of CLEB: LinINF (Exchangeable Hessian)

\[ D = [0, +\infty)^d, \quad F(x) = \sum_{i=1}^{d} \int_{0}^{x_i} \psi^{-1}(s)ds \]

\[ \psi(x) = \exp(\eta x) : \text{LinExp} \]
\[ \psi(x) = (\eta x)^{-q}, \quad q > 1 : \text{LinPoly} \]

\[ \text{INF, Audibert and Bubeck [2009]} \]
Different instances of CLEB: LinINF (Exchangeable Hessian)

\[ D = [0, +\infty)^d, \quad F(x) = \sum_{i=1}^{d} \int_{0}^{x_i} \psi^{-1}(s)ds \]

\[ \psi(x) = \exp(\eta x) : \text{LinExp} \]
\[ \psi(x) = (-\eta x)^{-q}, \quad q > 1 : \text{LinPoly} \]

INF, Audibert and Bubeck [2009]
Different instances of CLEB: Follow the regularized leader

\[ D = \text{Conv}(S), \text{ then} \]

\[ w_{t+1} \in \arg\min_{w \in D} \left( \sum_{s=1}^{t} \tilde{\ell}_s^T w + F(w) \right) \]

Strong connections with interior-point methods

 Particularly interesting choice: \( F \) self-concordant barrier function, Abernethy, Hazan and Rakhlin [2008]
$D = \text{Conv}(S)$, then

$$w_{t+1} \in \underset{w \in D}{\text{argmin}} \left( \sum_{s=1}^{t} \tilde{\ell}_s^T w + F(w) \right)$$

Strong connections with \textit{interior-point methods}

Particularly interesting choice: $F$ self-concordant barrier function, Abernethy, Hazan and Rakhlin [2008]
$\mathcal{D} = \text{Conv}(S)$, then

$$w_{t+1} \in \arg\min_{w \in \mathcal{D}} \left( \sum_{s=1}^{t} \tilde{\ell}_s^T w + F(w) \right)$$

Strong connections with interior-point methods

Particularly interesting choice: $F$ self-concordant barrier function, Abernethy, Hazan and Rakhlin [2008]
\[ \bar{R}_n = \inf_{\text{strategy}} \max_{S \subseteq \{0,1\}^d} \sup_{\text{adversaries}} R_n \]

**Theorem (Audibert, Bubeck and Lugosi [2011])**

Let \( n \geq d \). In the full information and semi-bandit games, we have:

\[
0.008 \ d \sqrt{n} \leq \bar{R}_n \leq d \sqrt{2n},
\]

and in the bandit game:

\[
0.01 \ d^{3/2} \sqrt{n} \leq \bar{R}_n \leq 2 \ d^{5/2} \sqrt{2n}.
\]
Minimax regret for combinatorial prediction games

\[ \overline{R}_n = \inf_{\text{strategy}} \max_{S \subset \{0,1\}^d} \sup_{\text{adversaries}} R_n \]

**Theorem (Audibert, Bubeck and Lugosi [2011])**

Let \( n \geq d \). In the full information and semi-bandit games, we have:

\[ 0.008 \, d \sqrt{n} \leq \overline{R}_n \leq d \sqrt{2n}, \]

and in the bandit game:

\[ 0.01 \, d^{3/2} \sqrt{n} \leq \overline{R}_n \leq 2 \, d^{5/2} \sqrt{2n}. \]
New project: Combinatorial testing
Set of concepts: $S \subset \{0, 1\}^d$

Paths

$k$-sets

$k$-sized intervals

Spanning trees
Set of concepts: $S \subset \{0, 1\}^d$

Paths

$k$-sets

$k$-sized intervals

Spanning trees
New project: Combinatorial testing

Set of concepts: $S \subset \{0, 1\}^d$

Paths

$k$-sets

$k$-sized intervals

Spanning trees
New project: Combinatorial testing

Set of concepts: $S \subset \{0, 1\}^d$

Paths

$k$-sets

$k$-sized intervals

Spanning trees
New project: Combinatorial testing

- Set of concepts: $\mathcal{S} \subset \{0, 1\}^d$
  - Paths
  - $k$-sets
  - $k$-sized intervals
  - Spanning trees

- Data: $X \in \mathbb{R}^d$

- Hypotheses:
  - $H_0$: “nothing special happens in $X$”
  - $H_1$: $\exists C \in \mathcal{S}$ s.t “something special happens on $X|_C$”
New project: Combinatorial testing

- Set of concepts: $S \subset \{0, 1\}^d$

Paths

$k$-sets

$k$-sized intervals

Spanning trees

- Data: $X \in \mathbb{R}^d$
- Hypotheses:

$H_0$ : “nothing special happens in $X$”

$H_1$ : $\exists C \in S$ s.t “something special happens on $X|_C$”
New project: Combinatorial testing

- Set of concepts: $S \subset \{0, 1\}^d$
- Paths

$k$-sets

$k$-sized intervals

Spanning trees

- Data: $X \in \mathbb{R}^d$
- Hypotheses:

$H_0$: “nothing special happens in $X$”

$H_1$: $\exists C \in S$ s.t “something special happens on $X|_C$”
Two examples of combinatorial testing problems

- Simultaneous tests: $|S| = 1$, Fan, Hall and Yao [2008]
- Detection of elevated mean:

  \[ H_0 : X \sim \mathcal{N}(0, I_d) \]
  \[ H_1 : \exists C \in S \text{ such that } X \sim \mathcal{N}(\mu 1_C, I_d) \]

For $k$-sets: problem suggested by Tukey, analyzed in Donoho and Jin [2002].
General framework introduced in Arias-Castro, Candès, Helgason and Zeitouni [2008].

- Detection of combinatorial correlation, Arias-Castro, Bubeck and Lugosi [2011]: $X_i \sim \mathcal{N}(0, 1), i \in \{1, \ldots, d\}$

  \[ H_0 : \mathbb{E}(X_i X_j) = 0 \]
  \[ H_1 : \exists C \in S \text{ such that } \mathbb{E}(X_i X_j) = \rho 1_{i \neq j, i, j \in C} \]
Two examples of combinatorial testing problems

- **Simultaneous tests:** $|S| = 1$, Fan, Hall and Yao [2008]
- **Detection of elevated mean:**

  \[
  H_0 : X \sim \mathcal{N}(0, I_d) \\
  H_1 : \exists C \in S \text{ such that } X \sim \mathcal{N}(\mu 1_C, I_d)
  \]

  For $k$-sets: problem suggested by Tukey, analyzed in Donoho and Jin [2002].
  General framework introduced in Arias-Castro, Candès, Helgason and Zeitouni [2008].

- **Detection of combinatorial correlation,** Arias-Castro, Bubeck and Lugosi [2011]: $X_i \sim \mathcal{N}(0, 1), \ i \in \{1, \ldots, d\}$

  \[
  H_0 : \mathbb{E}(X_i X_j) = 0 \\
  H_1 : \exists C \in S \text{ such that } \mathbb{E}(X_i X_j) = \rho 1_{i \neq j, i, j \in C}
  \]
Two examples of combinatorial testing problems

- **Simultaneous tests**: \(|S| = 1\), Fan, Hall and Yao [2008]
- **Detection of elevated mean**:

  \[
  H_0 : X \sim \mathcal{N}(0, I_d) \\
  H_1 : \exists C \in S \text{ such that } X \sim \mathcal{N}(\mu 1_C, I_d)
  \]

For \(k\)-sets: problem suggested by Tukey, analyzed in Donoho and Jin [2002].
General framework introduced in Arias-Castro, Candès, Helgason and Zeitouni [2008].

- **Detection of combinatorial correlation**, Arias-Castro, Bubeck and Lugosi [2011]: \(X_i \sim \mathcal{N}(0, 1), i \in \{1, \ldots, d\}\)

  \[
  H_0 : \mathbb{E}(X_iX_j) = 0 \\
  H_1 : \exists C \in S \text{ such that } \mathbb{E}(X_iX_j) = \rho 1_{i \neq j, i,j \in C}
  \]
few tests for detection of combinatorial correlation

\[ Z_C = X^T (A_C^{-1} - I_n) X, \quad (A_C)_{i,j} = \mathbb{1}_{i=j} + \rho \mathbb{1}_{i \neq j, i,j \in C} \]

- **Optimal test:** Likelihood ratio test

\[ \text{Reject if } \sum_{C \in S} \exp \left( -\frac{1}{2} Z_C \right) > \text{threshold} \]

- **Generalized Likelihood Ratio Test (GLRT):**

\[ \text{Reject if } \max_{C \in S} -\frac{1}{2} Z_C > \text{threshold} \]

- **Scan statistics:**

\[ \text{Reject if } \max_{C \in S} \sum_{i \neq j, i,j \in C} X_i X_j > \text{threshold} \]

- **Squared norm test:**

\[ \text{Reject if } ||X||_2 > \text{threshold} \]
Few tests for detection of combinatorial correlation

\[ Z_C = X^T (A_C^{-1} - I_n) X, \quad (A_C)_{i,j} = 1_{i=j} + \rho 1_{i \neq j, i,j \in C} \]

- **Optimal test:** Likelihood ratio test

  \[ \text{Reject if } \sum_{C \in S} \exp \left( -\frac{1}{2} Z_C \right) > \text{threshold} \]

- **Generalized Likelihood Ratio Test (GLRT):**

  \[ \text{Reject if } \max_{C \in S} -\frac{1}{2} Z_C > \text{threshold} \]

- **Scan statistics:**

  \[ \text{Reject if } \max_{C \in S} \sum_{i \neq j, i,j \in C} X_i X_j > \text{threshold} \]

- **Squared norm test:**

  \[ \text{Reject if } \|X\|_2 > \text{threshold} \]
Few tests for detection of combinatorial correlation

\[ Z_C = X^T (A_C^{-1} - I_n) X, \quad (A_C)_{i,j} = 1_{i=j} + \rho 1_{i\neq j, i,j \in C} \]

- **Optimal test: Likelihood ratio test**

  \[ \text{Reject if } \sum_{C \in S} \exp \left( -\frac{1}{2} Z_C \right) > \text{threshold} \]

- **Generalized Likelihood Ratio Test (GLRT):**

  \[ \text{Reject if } \max_{C \in S} -\frac{1}{2} Z_C > \text{threshold} \]

- **Scan statistics:**

  \[ \text{Reject if } \max_{C \in S} \sum_{i \neq j, i,j \in C} X_i X_j > \text{threshold} \]

- **Squared norm test:**

  \[ \text{Reject if } ||X||_2 > \text{threshold} \]
Few tests for detection of combinatorial correlation

\[ Z_C = X^T (A_C^{-1} - I_n) X, \quad (A_C)_{i,j} = 1_{i=j} + \rho 1_{i \neq j, i,j \in C} \]

- **Optimal test: Likelihood ratio test**
  
  \[ \text{Reject if } \sum_{C \in S} \exp \left( -\frac{1}{2} Z_C \right) > \text{threshold} \]

- **Generalized Likelihood Ratio Test (GLRT):**
  
  \[ \text{Reject if } \max_{C \in S} -\frac{1}{2} Z_C > \text{threshold} \]

- **Scan statistics:**
  
  \[ \text{Reject if } \max_{C \in S} \sum_{i \neq j, i,j \in C} X_i X_j > \text{threshold} \]

- **Squared norm test:**
  
  \[ \text{Reject if } \|X\|_2^2 > \text{threshold} \]
Few tests for detection of combinatorial correlation

\[ Z_C = X^T (A_C^{-1} - I_n) X, \quad (A_C)_{i,j} = 1_{i=j} + \rho 1_{i \neq j, i, j \in C} \]

- **Optimal test: Likelihood ratio test**

> Reject if \( \sum_{C \in S} \exp \left( -\frac{1}{2} Z_C \right) > \text{threshold} \)

- **Generalized Likelihood Ratio Test (GLRT):**

> Reject if \( \max_{C \in S} -\frac{1}{2} Z_C > \text{threshold} \)

- **Scan statistics:**

> Reject if \( \max_{C \in S} \sum_{i \neq j, i, j \in C} X_i X_j > \text{threshold} \)

- **Squared norm test:**

> Reject if \( \|X\|_2 > \text{threshold} \)
## Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>(k)-sized intervals</th>
<th>(k) sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal test</strong></td>
<td>Powerful if (\rho &lt;&lt; \frac{\log(d/k)}{k})</td>
<td>Conjecture: Powerful if (k &lt;&lt; \sqrt{d})</td>
</tr>
<tr>
<td><strong>GLRT</strong></td>
<td>Powerful if (\rho &gt;&gt; \frac{\log(d)}{k})</td>
<td>Conjecture: Powerful if (k &lt;&lt; d)</td>
</tr>
<tr>
<td><strong>Scan statistics</strong></td>
<td>-</td>
<td>Conjecture: Powerful if (k &lt;&lt; d)</td>
</tr>
<tr>
<td><strong>Squared norm test</strong></td>
<td>Powerful iff (\rho &gt;&gt; \frac{\sqrt{d}}{k})</td>
<td>Powerful iff (\rho &gt;&gt; \frac{\sqrt{d}}{k})</td>
</tr>
</tbody>
</table>
Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal test</strong></td>
<td>Powerful if $\rho &lt;&lt; \frac{\log(d/k)}{k}$</td>
<td>Conjecture: Powerful if $k &lt;&lt; \sqrt{d}$</td>
</tr>
<tr>
<td><strong>GLRT</strong></td>
<td>Powerful if $\rho &gt;&gt; \frac{\log(d)}{k}$</td>
<td>Conjecture: Powerful if $k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Scan statistics</strong></td>
<td>-</td>
<td>Conjecture: Powerful if $k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Squared norm test</strong></td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
</tr>
</tbody>
</table>
### Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal test</strong></td>
<td>Powerful if $\rho &lt;&lt; \frac{\log(d/k)}{k}$</td>
<td>Conjecture: Powerful if $k &lt;&lt; \sqrt{d}$</td>
</tr>
<tr>
<td><strong>GLRT</strong></td>
<td>Powerful if $\rho &gt;&gt; \frac{\log(d)}{k}$</td>
<td>Conjecture: Powerful if $k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Scan statistics</strong></td>
<td>-</td>
<td>Conjecture: Powerful if $k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Squared norm test</strong></td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
</tr>
</tbody>
</table>
### Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal test</strong></td>
<td>Powerless if ( \rho &lt;&lt; \frac{\log(d/k)}{k} )</td>
<td>Conjecture: Powerless if ( k &lt;&lt; \sqrt{d} )</td>
</tr>
<tr>
<td><strong>GLRT</strong></td>
<td>Powerful if ( \rho &gt;&gt; \frac{\log(d)}{k} )</td>
<td>Conjecture: Powerless if ( k &lt;&lt; d )</td>
</tr>
<tr>
<td><strong>Scan statistics</strong></td>
<td>-</td>
<td>Conjecture: Powerless if ( k &lt;&lt; d )</td>
</tr>
<tr>
<td><strong>Squared norm test</strong></td>
<td>Powerful iff ( \rho &gt;&gt; \frac{\sqrt{d}}{k} )</td>
<td>Powerful iff ( \rho &gt;&gt; \frac{\sqrt{d}}{k} )</td>
</tr>
</tbody>
</table>
Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal test</strong></td>
<td>Powerless if</td>
<td>Conjecture: Powerless if</td>
</tr>
<tr>
<td></td>
<td>$\rho &lt;&lt; \frac{\log(d/k)}{k}$</td>
<td>$k &lt;&lt; \sqrt{d}$</td>
</tr>
<tr>
<td><strong>GLRT</strong></td>
<td>Powerful if</td>
<td>Conjecture: Powerless if</td>
</tr>
<tr>
<td></td>
<td>$\rho &gt;&gt; \frac{\log(d)}{k}$</td>
<td>$k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Scan statistics</strong></td>
<td>-</td>
<td>Conjecture: Powerless if</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Squared norm test</strong></td>
<td>Powerful iff</td>
<td>Powerful iff</td>
</tr>
<tr>
<td></td>
<td>$\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
<td>$\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
</tr>
</tbody>
</table>
Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal test</td>
<td>Powerless if $\rho &lt;&lt; \frac{\log(d/k)}{k}$</td>
<td>Conjecture: Powerless if $k &lt;&lt; \sqrt{d}$</td>
</tr>
<tr>
<td>GLRT</td>
<td>Powerful if $\rho &gt;&gt; \frac{\log(d)}{k}$</td>
<td>Conjecture: Powerless if $k &lt;&lt; d$</td>
</tr>
<tr>
<td>Scan statistics</td>
<td>-</td>
<td>Conjecture: Powerless if $k &lt;&lt; d$</td>
</tr>
<tr>
<td>Squared norm test</td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
</tr>
</tbody>
</table>
Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal test</td>
<td>Powerful if $\rho &lt;&lt; \frac{\log(d/k)}{k}$</td>
<td>Conjecture: Powerful if $k &lt;&lt; \sqrt{d}$</td>
</tr>
<tr>
<td>GLRT</td>
<td>Powerful if $\rho &gt;&gt; \frac{\log(d)}{k}$</td>
<td>Conjecture: Powerful if $k &lt;&lt; d$</td>
</tr>
<tr>
<td>Scan statistics</td>
<td>-</td>
<td>Conjecture: Powerful if $k &lt;&lt; d$</td>
</tr>
<tr>
<td>Squared norm test</td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
<td>Powerful iff $\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
</tr>
</tbody>
</table>
Preliminary results for detection of combinatorial correlation

<table>
<thead>
<tr>
<th></th>
<th>$k$-sized intervals</th>
<th>$k$ sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal test</strong></td>
<td><strong>Powerless if</strong></td>
<td><strong>Conjecture: Powerless if</strong></td>
</tr>
<tr>
<td></td>
<td>$\rho &lt;&lt; \frac{\log(d/k)}{k}$</td>
<td>$k &lt;&lt; \sqrt{d}$</td>
</tr>
<tr>
<td><strong>GLRT</strong></td>
<td><strong>Powerful if</strong></td>
<td><strong>Conjecture: Powerless if</strong></td>
</tr>
<tr>
<td></td>
<td>$\rho &gt;&gt; \frac{\log(d)}{k}$</td>
<td>$k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Scan statistics</strong></td>
<td>-</td>
<td><strong>Conjecture: Powerless if</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k &lt;&lt; d$</td>
</tr>
<tr>
<td><strong>Squared norm test</strong></td>
<td><strong>Powerful iff</strong></td>
<td><strong>Powerful iff</strong></td>
</tr>
<tr>
<td></td>
<td>$\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
<td>$\rho &gt;&gt; \frac{\sqrt{d}}{k}$</td>
</tr>
</tbody>
</table>
Lots of unexplored extensions, both important for applications and mathematically elegant.
Lots of unexplored extensions, both important for applications and mathematically elegant.
Lots of unexplored extensions, both important for applications and mathematically elegant.

Detection of combinatorial correlation

Combinatorial LASSO?
Preprints

S. Bubeck and A. Slivkins, The best of both worlds: an adaptive strategy for stochastic and adversarial multi-armed bandits, submitted to COLT 2011

Journal Papers

J.Y. Audibert and S. Bubeck, Regret Bounds and Minimax Policies under Partial Monitoring, JMLR, 2010
S. Bubeck and U. von Luxburg, Nearest Neighbor Clustering: A Baseline Method for Consistent Clustering with Arbitrary Objective Functions, JMLR, 2009

Conference Papers (Acceptance ratio NIPS ~ 25%, COLT ~ 35%)

J.Y. Audibert, S. Bubeck and R. Munos, Best Arm Identification in Multi-Armed Bandits, COLT 2010
S. Bubeck and R. Munos, Open-Loop Optimistic Planning, COLT 2010
J.Y. Audibert and S. Bubeck, Minimax Policies for Adversarial and Stochastic Bandits, COLT 2009 (Best Student Paper Award)
S. Bubeck, R. Munos and G. Stoltz, Pure Exploration in Multi-Armed Bandit Problems, ALT 2009

PhD Thesis, Book Chapters, Technical Reports

S. Bubeck, Bandits Games and Clustering Foundations, PhD dissertation, 2010 (runner-up for the Gilles Kahn prize 2010)